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Multiplicity Result of Periodic Solutions for a Class of Damped Vibration Problems

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Abstract: A class of damped vibration problems (1) is studied by establishing a proper variational setting under certain conditions. A multiplicity result of periodic solutions of the damped vibration problems is obtained. The results presented in this paper imprave and extend some recent results of references [1] and references [2].
Key words: second order system; damped vibration problem; multiple periodic solutions; critical point
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一类阻尼振动问题的周期解的多重性结果

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摘要:研究了一类阻尼振动问题(1),通过建立恰当的变分设置,在一定条件下,获得了该阻尼振动问题的周期解的多重性结果.得到的结果推广和改进了文献[1]和文献[2]的主要结果.
关键词:二阶系统;阻尼振动问题;多重周期解;临界点

1 Introduction and Preliminaries

Consider the following damped vibration problems:

$$\begin{cases} \ddot{u}(t) + (q(t)I_{N\times N} + B)\dot{u}(t) + \nabla F(t,u(t)) = A(t)u(t) - \frac{1}{2}q(t)Bu(t), a.e.\ t \in [0,T], \\ u(0) - u(T) = \dot{u}(t) - \dot{u}(T) = 0. \end{cases} \tag{1}$$

Where $T > 0$, $q \in L^1(0,T;R)$ with $\int_0^T q(t)\,\mathrm{d}t = 0$, $A(t) = [a_{ij}(t)]$ is a symmetric $N \times N$ matrix-valued function defined in $[0,T]$ with $a_{ij} \in L^\infty([0,T])$ for all $i,j = 1,2,\cdots,N$ and there exists a positive constant θ such that $A(t)\xi \cdot \xi \geq \theta|\xi|^2$ for all $\xi \in R^N$ and $a.e.\ t \in [0,T]$, $B = [b_{ij}]$ is an antisymmetry $N \times N$ constant matrix, and $F: [0,T] \times R^N \to R$ satisfies the following assumption:

$(*)$ $F(t,x)$ is measurable in t for every $x \in R^N$ and continuously differentiable in x for $a.e.\ t \in [0,T]$, and there exist $p > 2$ and $a \in L^1(0,T;R^+)$ such that

$$|\nabla F(t,x)| \leq a(t)\,|x|^{p-1}. \tag{2}$$

For all $x \in R^N$ and $a.e.\ t \in [0,T]$.

For when B is a zero matrix, the damped vibration problems (1) were studied in [1], and an existence theorem and three multiplicity theorems of periodic solutions were given. For when $q(t) \equiv 0$ and $A(t)$ is a zero matrix, Fengjuan Meng and Fubao Zhang got some sufficient conditions for the existence for periodic solutions of system (1) in [2] by using the Minimax Theorem.

In the present paper, our main purpose is to study the existence of variational constructions for system (1), and as applications, we consider the multiplicity of periodic orbits for the system (1) via some critical point theorems.

This paper is organized as follows. In the end of this section we collect some critical point theorems which will be applied to our functional. We discuss a variational setting for system (1) in section 2. Our main results and proof

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will be given in the last section.

In order to state the critical point theorem which will be used to prove our main result, we need the following notions. Let X and Y be Banach spaces with X being separable and reflexive, and set $E = X \oplus Y$. Let $S \subset X^*$ be a dense subset. For each $s \in S$ there is a semi-norm on E defined by

$$p_s: E \to R, p_s(u) = |s(x)| + \|y\| \text{ for } u = x + y \in X \oplus Y.$$

We denote by T_S the topology on E induced by semi-norm family $\{p_s\}$, and let ω and ω^* denote the weak-topology and weak*-topology, respectively.

For a functional $\Phi \in C^1(E,R)$ we write $\Phi_a = \{u \in E: \Phi(u) \geq a\}$. Recall that Φ' is said to be weak sequentially continuous if for any $u_k \rightharpoonup u$ in E one has $\lim\limits_{k\to\infty} \Phi'(u_k)v = \Phi'(u)v$ for each $v \in E$, i.e. $\Phi': (E,\omega) \to (E^*,\omega^*)$ is sequentially continuous. For $c \in R$ we say that Φ satisfies the $(C)_c$ condition if any sequence $\{u_k\} \subset E$ contains a convergent subsequence, and such that $\Phi(u_k) \to c$ and $(1 + \|u_k\|)\Phi'(u_k) \to 0$ as $k \to \infty$.

Suppose that:

(Ⅰ) for any $c \in R$, Φ_c is T_S-closed, and $\Phi': (\Phi_c, T_S) \to (E^*, \omega^*)$ is continuous;

(Ⅱ) there exist $\rho > 0$ such that $\kappa := \inf \Phi(\partial B_\rho \cap Y) > 0$, where $B_\rho = \{u \in E: \|u\| < \rho\}$;

(Ⅲ) there exist a finite dimensional subspace $Y_0 \subset Y$ and $R > \rho$ such that $\bar{c} := \sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \backslash S_0) < \inf \Phi(B_\rho \cap Y)$, where $E_0 := X \oplus Y_0$, and $S_0 = \{u \in E_0: \|u\| \leq R\}$.

Theorem 1[3-4] Assume that Φ is even and (Ⅰ)—(Ⅲ) are satisfied. Then Φ has at least $m = \dim Y_0$ pairs of critical points with critical values less than or equal to $\bar{c}$ provided Φ satisfies the $(C)_c$ condition for all $c \in [\kappa, \bar{c}]$.

In our applications we take $S = X^*$, so that T_S is the product topology on $E = X \oplus Y$ given by the weak topology on X and the strong topology on Y. Moreover, we need the following lemma which can be found in [5].

Lemma 1 Suppose $\Phi \in C^1(E,R)$ be the norm

$$\Phi(u) = \frac{1}{2}(\|y\|^2 - \|x\|^2) - \Psi(u) \text{ for } u = x + y \in E = X \oplus Y,$$

such that:

1) $\Psi \in C^1(E,R)$ is bounded from below;

2) $\Psi: (E,\omega) \to R$ is sequentially lower semicontinuous, that is, $u_k \to u$ in (E,ω) implies $\Psi(u) \leq \liminf\limits_{k} \Psi(u_k)$;

3) $\Psi': (E,\omega) \to (E^*,\omega^*)$ is sequentially continuous;

4) $\nu: E \to R, \nu(u) = \|u\|^2$ is C^1 and $\nu': (E,\omega) \to (E^*,\omega^*)$ is sequentially continuous.

Then Φ satisfies (Ⅰ).

2 The Variational Principles

Let us have $H_T^1 = \{u: [0,T] \to R^N: u$ is absolutely continuous, $u(0) = u(t)$ and $\dot{u} \in L^2(0,T;R^N)\}$ with the inner product

$$\langle u,v\rangle_0 = \int_0^T (u(t),v(t))\,dt + \int_0^T (\dot{u}(t),\dot{v}(t))\,dt$$

for any $u,v \in H_T^1$, where $(\cdot,\cdot)$ denote the usual inner product in R^N. The corresponding norm is defined by

$$\|u\|_0 = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}$$

for $u \in H_T^1$. Then, obviously, H_T^1 is a Hilbert space.

In the paper, we always assume $Q(t) = \int_0^t q(s)\,ds$.

Set $$\|u\| = \left(\int_0^T e^{Q(t)}(A(t)u(t),u(t))\,dt + \int_0^T e^{Q(t)} |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}$$

for $u \in H_T^1$. Clearly, the norm $\|\cdot\|$ is equivalent to the usual one $\|\cdot\|_0$ on H_T^1. Denote by $\langle\cdot,\cdot\rangle$ the inner product corresponding to $\|\cdot\|$ on H_T^1. It is well known that H_T^1 is compactly embedded in $C(0,T;R^N)$ (for example, see Proposition 1.2 in [6]).

Defined the functional I on H_T^1 by

$$I(u) = \int_0^T e^{Q(t)}\left[\frac{1}{2}\,|\dot{u}(t)|^2 + \frac{1}{2}(Bu(t),\dot{u}(t)) + \frac{1}{2}(A(t)u(t),u(t)) - F(t,u(t))\right]\mathrm{d}t .$$

Then we have the following facts.

Lemma 2[7] The functional I is continuously differentiable on H_T^1.

Lemma 3[7] If $u \in H_T^1$ is a solution of the Euler equation $I'(u) = 0$, then u is a solution of problem (1).

Moreover, we need more preliminaries. We define an operator $L: H_T^1 \to H^*$ as follow, for any $u \in H_T^1$, which is given by

$$Lu(v) = \int_0^T e^{Q(t)}\left[(B\,\dot{u},v) + \frac{1}{2}q(t)(Bu,v)\right]\mathrm{d}t$$

for all $v \in H$, where H^* denotes the dual space of H_T^1. By Riesz representation theorem, we can identify H^* with H_T^1. Thus, Lu can also be viewed as function belonging to H_T^1 such that $\langle Lu,v\rangle = Lu(v)$ for any $u,v \in H_T^1$.

It is easy to check that L is a bounded linear operator on H_T^1. Moreover, L is self-adjoint on H_T^1 (see [7]).

Lemma 4[7] L is compact on H_T^1.

By the antisymmetry of B we see that

$$\langle Lu,u\rangle = \int_0^T e^{Q(t)}\left[(B\,\dot{u},u) + \frac{1}{2}q(t)(Bu,u)\right]\mathrm{d}t = \int_0^T e^{Q(t)}(B\,\dot{u},u)\,\mathrm{d}t .$$

Define $J(u) = \int_0^T e^{Q(t)}F(t,u(t))\,\mathrm{d}t$ for any $u \in H_T^1$. Then $I(u)$ can be rewritten as

$$I(u) = \frac{1}{2}\langle (I-L)u,u\rangle - J(u)\ , \tag{3}$$

where I denotes the identify operator. Using the classical spectral theory, we can decompose H_T^1 into the orthogonal sum of invariant subspaces for $(I-L)$

$$H_T^1 = H^- \oplus H^0 \oplus H^+\ ,$$

where $H^0 = \ker(I-L)$ and H^-, H^+ are such that, for some $\gamma > 0$,

$$\langle (I-L)u,u\rangle \leq -\gamma\ \|u\|^2 \tag{4}$$

for every $u \in H^-$, and

$$\langle (I-L)u,u\rangle \geq \gamma\ \|u\|^2 \tag{5}$$

for every $u \in H^+$.

Furthermore, L has only finitely many eigenvalues λ_i with $\lambda_i > 1$ since L is compact on H_T^1. Hence H^- is finite dimensional. Notice that $(I-L)$ is a compact perturbation of the self-adjoint operator I. By the spectral theory of compact operator, we know that 0 is not in the essential spectrum of $(I-L)$. Hence H^0 is a finite dimensional space too.

By (4), (5) and the boundedness of $(I-L)$, we can define another equivalent norm $\|\cdot\|_*$ on H_T^1 given by

$$\|u\|_* = \left(\langle (I-L)u^+,u^+\rangle - \langle (I-L)u^-,u^-\rangle + \langle u^0,u^0\rangle\right)^{\frac{1}{2}}$$

where $u^+ \in H^+, u^- \in H^-, u^0 \in H^0$ with $u = u^+ + u^- + u^0$. This yields

$$I(u) = \frac{1}{2}\left(\|u^+\|_*^2 - \|u^-\|_*^2\right) - J(u)$$

for $u = u^+ + u^- + u^0 \in H^+ \oplus H^- \oplus H^0$.

Denote by $\langle\cdot,\cdot\rangle_*$ the inner products corresponding to $\|\cdot\|_*$ on H_T^1. Then the above argument shows that the spaces H^+, H^- and H^0 are mutually orthogonal with respect to the inner products $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_*$.

Lemma 5 $J': (H_T^1,\omega) \to ((H_T^1)^*,\omega^*)$ is sequentially continuous under the assumption $(*)$, that is, $u_k \rightharpoonup u$ in H_T^1 implies $J'(u_k) \rightharpoonup J'(u)$.

Proof Let $\{u_k\} \subset H_T^1$ be any sequence converging to some u weakly. Proposition 1.2 in [6] implies that $\{u_k\}$ converges uniformly to u on $[0,T]$. Hence $(\nabla F(t,u_k(t)),v(t)) \to (\nabla F(t,u(t)),v(t))\ a.e.$ on $[0,T]$ for each $v \in H_T^1$. It follows from assumption $(*)$ that

$$|(\nabla F(t,u_k),v)| \leq a(t)\ \|u_k\|_\infty^{p-1}|v| \to a(t)\ \|u\|_\infty^{p-1}|v|$$

for all $v \in H_T^1$. By the Lebesgue convergence theorem, one has

$$\int_0^T e^{Q(t)}(\nabla F(t,u_k),v)\,dt \to \int_0^T e^{Q(t)}(\nabla F(t,u),v)\,dt$$

for all $v \in H_T^1$, that is, $J'(u_k) \rightharpoonup J'(u)$. The proof is completed.

3 Main Results

In this section we always assume that c and c_i stand for different positive constants.

Theorem 2 Assume that $F \in C^1([0,T]\times R^N, R)$ satisfies ($*$) and the following conditions:

1) $F(t,x)$ is even in x and $F(t,0) = 0$;

2) there exists a $N \times N$ symmetric matrix $D(t) = [d_{ij}(t)]$ such that $\lim\limits_{|x|\to\infty} \dfrac{|\nabla F(t,x) - D(t)x|}{|x|} = 0$ uniformly in $t \in [0,T]$, where $d_{ij}(t) \in L^\infty([0,T])$ for all $i,j = 1,2,\cdots,N$;

3) there exists $\delta > 0$ such that $(D(t)x,x) \geq \beta\ |x|^2$ for $a.e.$ $t \in [0,T]$ and $x \in R^N$, where $\beta = (\delta + \max\limits_{1\le i,j\le N}\ \operatorname*{esssup}\limits_{0\le t\le T}|a_{ij}(t)|)e^{2\|q\|_{L^1}}$;

4) $\hat{F}(t,x) \geq 0, \hat{F}(t,x) \to +\infty$ as $|x| \to \infty$ uniformly in $t \in [0,T]$, where $\hat{F}(t,x) = \dfrac{1}{2}(\nabla F(t,x),x) - F(t,x)$;

5) $\ker(I-L) = 0$, where $(I-L)$ defined as in section 2.

Then (1) has at least N pairs T-periodic solutions.

Proof It follows from 5) that $H_T^1 = H^+ \oplus H^-$, where H^+, H^- defined as in section 2. Let $E = H_T^1, X = H^-$, $Y = H^+, \Phi(u) = I(u)$ and $\Psi(u) = J(u)$. We will show that Φ satisfies all hypotheses of Theorem 1.

First, we check that Φ satisfies (I).

By Lemma 2 and Lemma 5, we see that $\Psi(u) \in C^1(E,R)$ satisfies 3) of Lemma 1. It is pointed out in [6] that Ψ is weakly continuous under the assumption ($*$), that is, Ψ satisfies 2) of Lemma 1. Moreover, note that E is a Hilbert space. Hence 4) of Lemma 1 holds, obviously.

It remains to prove that Ψ is bounded from blow. By 2) there exist a constant $c > 0$ such that

$$|\nabla F(t,x) - D(t)x| \le \frac{\beta}{2}(|x| + c) \tag{6}$$

for $a.e.$ $t \in [0,T]$ and all $x \in R^N$. Consequently, by (6), 1) and 3), one has

$$\begin{aligned}
\Psi(u) &= \int_0^T e^{Q(t)} F(t,u)\,dt \\
&= \int_0^T\int_0^1 e^{Q(t)}(\nabla F(t,su),u)\,ds\,dt \\
&= \frac{1}{2}\int_0^T e^{Q(t)}(D(t)u,u)\,dt + \int_0^T\int_0^1 e^{Q(t)}(\nabla F(t,su) - D(t)(su),u)\,ds\,dt \\
&\geq \frac{\beta}{2}\int_0^T e^{Q(t)}\ |u|^2 dt - \int_0^T\int_0^1 e^{Q(t)}[\frac{\beta}{2}(|su| + c)|u|]\,ds\,dt \\
&= \frac{\beta}{4}\int_0^T e^{Q(t)}[|u|^2 - 2c|u|]\,dt,
\end{aligned}$$

which implies that Ψ is bounded from blow. By virtue of lemma 1, Φ satisfies (I).

Furthermore, for $u \in Y$, by (2) we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\ \|u\|_*^2 - \int_0^T e^{Q(t)} F(t,u)\,dt \\
&\geq \frac{1}{2}\ \|u\|_*^2 - \int_0^T a(t)e^{Q(t)}\ |u|^p dt \\
&\geq \frac{1}{2}\ \|u\|_*^2 - \ \|u\|_\infty^p e^{\|q\|_{L^1}} \int_0^T a(t)\,dt \\
&\geq \frac{1}{2}\ \|u\|_*^2 - c_1\ \|u\|_*^p .
\end{aligned}$$

Since $p > 2$, there is small $\rho > 0$ such that $\frac{1}{4}\rho^2 \geq c_1\rho^p$. Therefore,

$$\kappa := \inf \Phi(\partial B_\rho \cap Y) \geq \frac{1}{4}\rho^2 > 0 \tag{7}$$

and hence (II) holds.

Next, we prove that (III) is satisfies under the hypotheses of Theorem 2.

Letting $Y_0 = R^N$, one can easily get that $Y_0 \subset Y$ and $\dim Y_0 = N$. In order to obtain the desired conclusion, it is sufficient to prove that $\Phi(u) \to -\infty$ as $\|u\| \to \infty$ in $E_0 = X \oplus Y_0$.

Let $\widetilde{F}(t,x) = F(t,x) - \frac{1}{2}(D(t)x,x)$ and $\delta_0 = \max\limits_{1\le i,j\le N} \operatorname*{esssup}\limits_{0\le t\le T} |a_{ij}(t)|$. Then for $u = x + y \in E_0$ we see that

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}(\|y\|_*^2 - \|x\|_*^2) - \int_0^T e^{Q(t)} F(t,u)\mathrm{d}t \\
&= \frac{1}{2}\langle (I-L)y,y\rangle - \frac{1}{2}\|x\|_*^2 - \frac{1}{2}\int_0^T e^{Q(t)}(D(t)u,u)\mathrm{d}t - \int_0^T e^{Q(t)}\widetilde{F}(t,u)\mathrm{d}t \\
&= \frac{1}{2}\int_0^T e^{Q(t)}(A(t)y,y)\mathrm{d}t - \frac{1}{2}\|x\|_*^2 - \frac{1}{2}\int_0^T e^{Q(t)}(D(t)u,u)\mathrm{d}t - \int_0^T e^{Q(t)}\widetilde{F}(t,u)\mathrm{d}t \\
&\leqslant \frac{\delta_0}{2}e^{\|q\|_{L^1}}\|y\|_{L^2}^2 - \frac{1}{2}\|x\|_*^2 - \frac{\delta+\delta_0}{2}e^{\|q\|_{L^1}}\|u\|_{L^2}^2 - \int_0^T e^{Q(t)}\widetilde{F}(t,u)\mathrm{d}t \\
&= \frac{\delta_0}{2}e^{\|q\|_{L^1}}\|y\|_{L^2}^2 - \frac{1}{2}\|x\|_*^2 - \frac{\delta+\delta_0}{2}e^{\|q\|_{L^1}}(\|x\|_{L^2}^2 + \|y\|_{L^2}^2) - \int_0^T e^{Q(t)}\widetilde{F}(t,u)\mathrm{d}t \\
&\leqslant -\frac{\delta}{2}e^{\|q\|_{L^1}}\|y\|_{L^2}^2 - \frac{1}{2}\|x\|_*^2 - \int_0^T e^{Q(t)}\widetilde{F}(t,u)\mathrm{d}t \\
&\leqslant -(\min\{\frac{\delta}{2\delta_0},\frac{1}{2}\})\|u\|_*^2 - \int_0^T e^{Q(t)}\widetilde{F}(t,u)\mathrm{d}t.
\end{aligned}$$

It remains to show that

$$\frac{1}{\|u\|^2}\int_0^T e^{Q(t)}\widetilde{F}(t,u)\mathrm{d}t \to 0 \tag{8}$$

as $\|u\| \to \infty$ in E_0. Indeed, for any $\varepsilon > 0$, by (2) and 2) we know that there exists positive constant $c = c(\varepsilon)$ such that

$$|\nabla F(t,x) - D(t)x| \le \varepsilon|x| + c \tag{9}$$

for $a.e.$ $t \in [0,T]$ and all $x \in R^N$. Thus, for $u \in E$ with $\|u\| \neq 0$, we have

$$\begin{aligned}
\left|\frac{1}{\|u\|^2}\int_0^T e^{Q(t)}\widetilde{F}(t,u)\mathrm{d}t\right| &= \left|\frac{1}{\|u\|^2}\int_0^T\int_0^1 e^{Q(t)}(\nabla F(t,su) - sD(t)u,u)\mathrm{d}s\mathrm{d}t\right| \\
&\leqslant \frac{e^{\|q\|_{L^1}}}{\|u\|^2}\int_0^T\int_0^1(\varepsilon|su| + c)|u|\mathrm{d}s\mathrm{d}t \\
&\leqslant \frac{e^{\|q\|_{L^1}}}{\|u\|^2}(\varepsilon\|u\|_{L^2}^2 + c\|u\|_{L^1}) \\
&\leqslant c_2(\varepsilon + \frac{c}{\|u\|}),
\end{aligned}$$

which implies that (8) is true provided the arbitrariness of ε. Hence, (III) holds.

Finally, it is remained to prove that Φ satisfies the $(C)_c$ condition for any $c \in R$. We assume that $\{u_k\} \subset E$ is any sequence such that

$$\Phi(u_k) \to c,\ (1 + \|u_k\|_*)\Phi'(u_k) \to 0 \tag{10}$$

as $k \to \infty$. Then we claim that $\{u_k\}$ is bounded in E. Assume by contradiction that $\|u_k\|_* \to \infty$ as $k \to \infty$. Let $\varphi_k = \frac{u_k}{\|u_k\|_*}$. Then $\|\varphi_k\|_* = 1$. Without loss of generality, we can assume that $\varphi_k \rightharpoonup \varphi$ in E and $\varphi_k \to \varphi$ in $C(0,T;R^N)$.

If $\varphi \neq 0$, set $\Omega = \{t \in [0,T] : \varphi(t) \neq 0\}$. Then Ω has a positive measure and $u_k(t) \to \infty$ for all $t \in \Omega$. It follows from 4) and (10) that

$$\begin{aligned}
c &= \lim_{k\to\infty}[\Phi(u_k) - \frac{1}{2}\Phi'(u_k)u_k] \\
&= \lim_{k\to\infty}\int_0^T e^{Q(t)}[\frac{1}{2}(\nabla F(t,u_k),u_k) - F(t,u_k)]\mathrm{d}t \\
&\geqslant \int_\Omega \liminf_{k\to\infty} e^{Q(t)}\hat{F}(t,u_k)\mathrm{d}t \to +\infty.
\end{aligned}$$

This is a contradiction. Therefore the case $\varphi \neq 0$ cannot occur, and hence $\varphi \equiv 0$. By (9) one has

$$
\begin{aligned}
&\left|\frac{1}{\|u_k\|_*}\int_0^T e^{Q(t)}(\nabla F(t,u_k),\varphi_k^+)\,\mathrm{d}t\right| \\
&\leqslant \left|\frac{1}{\|u_k\|_*}\int_0^T e^{Q(t)}[(\nabla F(t,u_k),\varphi_k^+)-(D(t)u_k,\varphi_k^+)]\,\mathrm{d}t\right| + \left|\int_0^T e^{Q(t)}(D(t)\varphi_k,\varphi_k^+)\,\mathrm{d}t\right| \\
&\leqslant \frac{1}{\|u_k\|_*}\int_0^T e^{Q(t)}(\varepsilon|u_k|+c)|\varphi_k^+|\,\mathrm{d}t + e^{\|q\|_{L^1}}(\max_{1\le i,j\le N}\operatorname{esssup}_{0\leqslant t\leqslant T}|d_{ij}(t)|)\int_0^T|\varphi_k||\varphi_k^+|\,\mathrm{d}t \\
&\leqslant \frac{e^{\|q\|_{L^1}}}{\|u_k\|_*}\|\varphi_k^+\|_\infty T(\varepsilon\|u_k\|_\infty+c)+c_3\|\varphi_k\|_\infty\|\varphi_k^+\|_\infty \\
&\leqslant c_4\left[\frac{\|\varphi_k^+\|_*}{\|u_k\|_*}(\varepsilon\|u_k\|_*+c)+\|\varphi_k\|_\infty\|\varphi_k^+\|_*\right] \\
&\leqslant c_4\left(\varepsilon+\frac{c}{\|u_k\|_*}+\|\varphi_k\|_\infty\right).
\end{aligned}
$$

By $\|u_k\|_*\to\infty$, $\|\varphi_k\|_\infty\to 0$ and the arbitrariness of ε, we get that

$$\frac{1}{\|u_k\|_*}\int_0^T e^{Q(t)}(\nabla F(t,u_k),\varphi_k^+)\,\mathrm{d}t\to 0 \tag{11}$$

as $k\to\infty$. Consequently,

$$
\begin{aligned}
o(1) &= \frac{\Phi'(u_k)\varphi_k^+}{\|u_k\|_*} \\
&= \langle (I-L)\varphi_k,\varphi_k^+\rangle - \frac{1}{\|u_k\|_*}\int_0^T e^{Q(t)}(\nabla F(t,u_k),\varphi_k^+)\,\mathrm{d}t \\
&= \|\varphi_k^+\|_*^2 + o(1).
\end{aligned}
$$

This yields $\|\varphi_k^+\|_*\to 0$, similarly, $\|\varphi_k^-\|_*^2\to 0$. This is also a contradiction since $\|\varphi_k\|_* = 1$ for any k. Therefore $\{u_k\}$ is bounded in E. Going if necessary to a subsequence, we can assume that $u_k\rightharpoonup u$ in H_T^1 and $\|u_k-u\|_\infty\to 0$. Since

$$
\begin{aligned}
\|u_k-u\|^2 &= \langle \Phi'(u_k)-\Phi'(u),u_k-u\rangle + \frac{1}{2}\int_0^T e^{Q(t)}q(t)(B(u_k-u),u_k-u)\,\mathrm{d}t \\
&\quad + \int_0^T e^{Q(t)}(B(\dot u_k-\dot u),u_k-u)\,\mathrm{d}t + \int_0^T e^{Q(t)}(\nabla F(t,u_k)-\nabla F(t,u),u_k-u)\,\mathrm{d}t \\
&\leqslant \langle \Phi'(u_k)-\Phi'(u),u_k-u\rangle + c_5\int_0^T|\dot u_k-\dot u||u_k-u|\,\mathrm{d}t \\
&\quad + \int_0^T e^{Q(t)}(|\nabla F(t,u_k)|+|\nabla F(t,u)|)|u_k-u|\,\mathrm{d}t \\
&\leqslant \langle \Phi'(u_k)-\Phi'(u),u_k-u\rangle + c_5\|u_k-u\|_\infty\int_0^T(|\dot u_k|+|\dot u|)\,\mathrm{d}t \\
&\quad + \|u_k-u\|_\infty\int_0^T e^{Q(t)}a(t)(|u_k|^{p-1}+|u|^{p-1})\,\mathrm{d}t,
\end{aligned}
$$

we see that $u_k\to u$ in H_T^1, i. e. Φ satisfies the $(C)_c$ condition for any $c\in R$. Note that 1) implies that Φ is even. Now the conclusion of Theorem 2 follows from Theorem 1. The proof is completed.

Remark 1 Even in the case that $q(t)\equiv 0$, and both $A(t)$ and B are zero matrices, Theorem 2 is new too.

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