昆明学院学报 2012, 34(3):60~63 CN 53-1211/G4 ISSN 1674-5639
Journal of Kunming University

The Equivalent Relations between the Continuity of Duality Mappings and some Properties of the η-Rotundity Modulus

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Abstract: Some properties and forms for the continuity of the duality mapping are discussed. The relations between them and the continuity of the supporting function coming from the geometric theory of Banach space are aroused. The equivalent relations between the continuity of the duality mapping and some properties for the η-rotundity modulus in some subsets are proved.

Key words: Banach space; duality mapping; η-Rotundity modulus; supporting function; rotundity norms
CIC number: O177.2 **Document code**: A **Article ID**: 1674-5639(2012)03-0060-04

对偶映像连续性和 η-凸性模的若干性质及其等价关系

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摘要:讨论了 Bananch 空间中对偶映像连续性的若干形式及其与 Banach 空间几何理论中支撑函数间的关系,以及 η-凸性模在某些子集上的性质. 证明了对偶映像连续性和 η-凸性模在这些子集上性质的等价关系.

关键词:Banach 空间;对偶映像;η-凸性模;支撑函数;凸性范数.

1 Introduction

Let X be a Banach space, X^* and X^{**} be its conjugate space and the *sec*ondary conjugate space respectively. $\forall x \in X$, denote

$$F(x) = \left\{ x^* \in X^* \mid \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\},$$

then $F(x)$ is a set-valued mapping from $X \to 2^{X^*}$, we call it the duality mapping on X. For a long time, there have been a lot of discussions and applications on the duality mapping in the fields of functional analysis and nonlinearity analysis. In usual documents, for example in the paper[1~2], the definitions for the continuity of the duality mapping were expressed as follows: for $x_0 \in X$, $F(x)$ is said to be norm-norm continuous at x_0, if $\forall \varepsilon > 0$, there exists $\delta > 0$, such that $\forall x \in X$, $\|x - x_0\| < \delta$ implies $\|F(x) - F(x_0)\| < \varepsilon$ (or, $x \to x_0$ implies $F(x) \to F(x_0)$); $F(x)$ is said to be norm-weak continuous at x_0, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall x \in X$, $\|x - x_0\| < \delta$ implies $\forall y^{**} \in X^{**}$, $\| \langle F(x), y^{**} \rangle - \langle F(x_0), y^{**} \rangle \| < \varepsilon$, (or, $x \to x_0$ implies $F(x) \xrightarrow{w} F(x_0)$); $F(x)$ is said to be norm-*weak** continuous at x_0, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall x \in X$, $\|x - x_0\| < \delta$ implies $\forall y \in X$, $\| \langle y, F(x) \rangle - \langle y, F(x_0) \rangle \| < \varepsilon$, (or, $x \to x_0$ implies $F(x) \xrightarrow{w^*} F(x_0)$); $F(x)$ is said to be norm-norm uniformly continuous on $D \subset X$, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall x, y \in D$, $\|x - y\| < \delta$ implies $\|F(x) - F(y)\| < \varepsilon$.

Furthermore, $F(x)$ is continuous, weak continuous and *weak** continuous on X, means that $F(x)$ is continuous, weak continuous and *weak** continuous at every point in X. It is easy to check that for the duality mapping $F(x)$, the following relations hold: continuity leads to weak continuity; weak continuity leads to *weak** continuity; $F(x)$ is uniformly continuous on X leads to $F(x)$ is continuous on X.

Noticing that the duality mapping $F(x)$ is a set-valued mapping, in usual cases, $F(x)$ is a set, may not be single-valued, so the descriptions about the continuity of the duality mapping above are not so convenient in some cases, and sometimes during the process of deduction there even exist some problems on the strictness. Till now, few papers paid enough attention to the relations between the continuity of the duality mapping and the supporting func-

收稿日期:2012-04-22
基金项目:昆明学院科研课题资助项目(XJ11L020)
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tion which comes from the geometric theory of Banach space, which made the work complex. For example, in[2] the writer proved the equivalent relations between the continuity of the duality mapping and the rotundity in Banach space at a length. In fact, as the rotundity could be described by the continuity of the supporting function in the geometric theory of Banach space, through which the proof in[2] can be easier.

The purpose of the paper is to discuss the relations between the non-negativity of the η-Rotundity modulus on some subsets and the rotundity norms in Banach spaces. By the result, the equivalent relations between the continuity of the duality mapping and the non-negativity of the η-Rotundity modulus on some subsets are proved.

2 Properties for the Continuity of Duality Mappings

We first introduce the definition of the supporting function.

Definition 1[3] Let X be a Banach space, $S=\{x \mid x\in X, \|x\|=1\}$ be the unit sphere of X, if $\forall x\in S$, there exist $x^*\in X^*$, such that the mapping $f_x: x\to x^*$, $<x, f_x>=f_x(x)=x^*$ satisfies $<x, f_x>=1=\|f_x\|$, then f_x is said to be the supporting function on the unit sphere S. If for $\forall x\in X$, there exist $t\in S$ and real number $\lambda\geqslant 0$ such that $x=\lambda t$, denote $f_x=f_{\lambda t}=\lambda f_t$, and then f_x is said to be the supporting function on X.

The continuity of the supporting function f_x are described as follows[3]: for $x_0\in X$, f_x is said to be continuous at x_0, if $x\to x_0$ implies $f_x\to f_{x_0}$; f_x is said to be weak-continuous at x_0, if $x\to x_0$ and for $\forall y^{**}\in X^{**}$, $<f_x, y^{**}>\to <f_{x_0}, y^{**}>$; f_x is said to be *weak**-continuous at x_0, if $x\to x_0$ implies that for $\forall y\in X$, $f_x(y)\to f_{x_0}(y)$; f_x is said to be uniformly continuous in a subset $D\subset X$, if $\forall \varepsilon>0$, $\exists \delta>0$, for $\forall x, y\in D$, as $\|x-y\|<\delta$, $\|f_x-f_y\|<\varepsilon$.

Lemma 1 Let X be a Banach space, f_x and $F(x)$ are the supporting function and the duality mapping respectively, Then for $\forall x\in X$, $f_x\in F(x)$. Furthermore $F(x)$ is single-valued at x if one of the following assumptions is satisfied:

1) f_x is the unique function in $F(x)$;

2) $F(x)$ is norm-*weak** lower semi-continuous at x. (the definition see[3]);

3) $F(x)$ is norm-*weak** continuous at x;

4) f_x is norm-*weak** continuous at x.

Proof Since f_x is the supporting function on the unit sphere S, for $\forall x\in S$, $\|x\|=1$ and $<x, f_x>=1=\|f_x\|$, we have $f_x(x)=<x, f_x>=1=\|x\|^2=\|f_x\|^2$, so $f_x\in F(x)$.

For $x\in X$, denote $x=\lambda t$ ($t\in S$, $\lambda\geqslant 0$, then $f_t(t)=\|t\|^2=\|f_t\|^2$, by the simple deduction, we have

$$f_x(x)=f_{\lambda t}(\lambda t)=\lambda^2 f_t(t)=\lambda^2\|t\|=\lambda^2\|t\|^2=\|\lambda t\|^2=\|x\|^2, \tag{1}$$

and

$$f_x(x)=f_{\lambda t}(\lambda t)=\lambda^2\|f_t\|=\lambda^2\|f_t\|^2=\|\lambda f_t\|^2=\|f_{\lambda t}\|^2=\|f_x\|^2, \tag{2}$$

together with (1) and (2), we have $f_x(x)=\|x\|^2=\|f_x\|^2$, so $f_x\in F(x)$.

1) Since $f_x\in F(x)$, if f_x is the unique function in $F(x)$, then $f_x=F(x)$, obviously, $F(x)$ is single-valued;

2) Since the duality mapping $F(x)$ is the sub-differential $\partial\phi(x)$ of $\phi(x)=\frac{1}{2}\|x\|^2$, and the sub-differential $\partial\phi(x)$ is single-valued at x if and only if it is *weak** lower-semi-continuous at x (see[4]);

3) It is well known. See[1,4];

4) If f_x is norm-*weak** continuous at x, but $F(x)$ is not single-valued at x. Suppose that there exist two differential functions $f_x, \tilde{f}_x\in F(x)$. For a positive real sequence $\lambda_n\to 0$, for $\forall y\in X$, when $x+\lambda_n y\to x$ ($n\to\infty$), $f_{x+\lambda_n y}\xrightarrow{w^*} f_x$ and $f_{x+\lambda_n y}\xrightarrow{w^*}\tilde{f}_x$, then we have

$$\begin{aligned}\frac{1}{2}\|x+\lambda_n y\|^2-\frac{1}{2}\|x\|^2 &= \|x+\lambda_n y\|^2-\frac{1}{2}(\|x+\lambda_n y\|^2+\|x\|^2)\leqslant \|x+\lambda_n y\|^2-(\|x+\lambda_n y\|\|x\|)\\ &\leqslant f_{x+\lambda_n y}(x+\lambda_n y)-f_{x+\lambda_n y}(x)\leqslant f_{x+\lambda_n y}[(x+\lambda_n y)-x]=f_{x+\lambda_n y}(\lambda_n y)\\ &=\lambda_n f_{x+\lambda_n y}(y).\end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned}\frac{1}{2}\|x\|^2-\frac{1}{2}\|x+\lambda_n y\|^2 &= \|x\|^2-\frac{1}{2}(\|x\|^2+\|x+\lambda_n y\|^2)\leqslant \|x\|^2-(\|x\|\|x+\lambda_n y\|)\\ &\leqslant \tilde{f}_x(x)-\tilde{f}_x(x+\lambda_n y)=\tilde{f}_x[x-(x+\lambda_n y)]=\tilde{f}_x(-\lambda_n y)\\ &=-\lambda_n\tilde{f}_x(y),\end{aligned} \tag{4}$$

(3) and (4) gives $0\leqslant \lambda_n f_{x+\lambda_n y}(y)-\lambda_n\tilde{f}_x(y)$ and $\tilde{f}_x(y)\leqslant f_{x+\lambda_n y}(y)$.

Let $n\to\infty$, since $x+\lambda_n y\to x$ and $f_{x+\lambda_n y}\xrightarrow{w^*} f_x$, then for $\forall y\in X$, we have

$$\tilde{f}_x(y)\leqslant f_x(y). \tag{5}$$

Employing the same method above, we obtain

$$f_x(y) \leqslant \tilde{f}_x(y). \tag{6}$$

By (5) and (6), for $\forall y \in X$, we have $f_x(y) = \tilde{f}_x(y)$. It shows that

$$f_x = \tilde{f}_x, \tag{7}$$

it is a contradiction to the assumption, so $F(x)$ is single-valued at x.

In the following, some equivalent conditions for the continuity of the duality mapping are discussed.

Theorem 1 Let X be a Banach space, X^* be the conjugate space of X, $F(x)$ be the duality mapping on X. Then the following conditions in every group are equivalent:

1) (1 − 1) $F(x)$ is norm-norm continuous on X;

(1 − 2) $\forall x \in X, x_n \to x, x^* \in F(x)$, then $\exists x_n^* \in F(x_n)$, such that $x_n^* \to x^*$ $(n \to \infty)$;

(1 − 3) For any $x \in X$, there exists a supporting function $f_x: X \to X^*$, which is norm-norm continuous on X.

2) (2 − 1) $F(x)$ is norm-weak continuous on X;

(2 − 2) $\forall x \in X, x_n \to x, x^\circ \in F(x)$, then $\exists x_n^* \in F(x_n)$, such that $\forall y^{**} \in X^{**}$, $< x_n^*, y^{**} > \to < x^\circ, y^{**} >$ $(n \to \infty)$. (also denote: $x_n^* \xrightarrow{w} x^*$);

(2 − 3) For each $x \in X$, there exists a supporting function $f_x: X \to X^*$, which is norm-weak continuous on X.

3) (3 − 1) $F(x)$ is norm-*weak** continuous on X;

(3 − 2) $\forall x \in X, x_n \to x, x^* \in F(x)$, then $\exists x_n^* \in F(x_n)$, such that $\forall y \in X$, $< y, x_n^* > \to < y, x^* >$ $(n \to \infty)$. (also denote: $x_n^* \xrightarrow{w^*} x^*$);

(3 − 3) For any $x \in X$, there exists a supporting function $f_x: X \to X^*$, which is norm-*weak** continuous on X.

4) (4 − 1) $F(x)$ is norm-norm uniformly continuity on $D \subset X$;

(4 − 2) $\forall \varepsilon > 0$, $\exists \delta > 0$, for $\forall x, y \in D$, $\|x - y\| < \delta$, then $\exists x^* \in F(x)$ and $\exists y^* \in F(y)$, such that $\|x^* - y^*\| < \varepsilon$;

(4 − 3) For any $x \in X$, there exists a supporting function $f_x: X \to X^*$, which is norm-norm uniformly continuity on $D \subset X$.

Proof We first prove group 1):

If (1 − 1) holds, which means that $F(x)$ is norm-norm continuous on X, then we have $F(x)$ is norm-*weak** continuous on X, by Lemma1, it follows that: $F(x)$ is single-valued on X, which means that $F(x) = x^*$, $F(x_n) = x_n^*$, since $F(x_n) \to F(x)$ $(n \to \infty)$, we have $x_n^* \to x^{\circ *}$ $(n \to \infty)$, so (1 − 2) holds.

Conversely, if (1 − 2) holds, then $x_n^* \to x^*$ $(n \to \infty)$ implies $x_n^* \xrightarrow{w} x^*$. The result in Lemma1 gives that $F(x)$ is single-valued on X, considering the assumption of $x_n^* \to x^*$ $(n \to \infty)$, we have $F(x_n) \to F(x)$ $(n \to \infty)$, which means (1 − 1) holds.

Taking the supporting function $f_x \in F(x)$ into account, according to Lemma1 and the proof above, we get (1 − 1) is equivalent to (1 − 3).

By applying the above technique ana*log*ously, 2), 3) and 4) is proved.

3 The Relations Between the Continuity of the Duality Mapping and the η-Rotundity Modulus

Denote A as the nonempty subset in $X \backslash \{\theta\}$, M the nonempty bounded subset in X, and

$$\eta(A, M) = \inf_{z \in A} \left\{ \inf_{\substack{x \in M \\ y \in X \\ x - y = z}} \left[\frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \left\| \frac{x+y}{2} \right\|^2 \right] \right\},$$

$\eta(A, M)$ is said to be the η-Rotundity modulus on the nonempty subset A in X concerning the bounded nonempty subset M (see[5]).

We first recall the definitions in[3]. Let X be a Banach space, $S = \{x \mid x \in X, \|x\| = 1\}$ be its unit sphere. X is said to be uniformly rotund, if for $\forall x_n, y_n \in S$, such that $\|x_n + y_n\| \to 2$, then $x_n - y_n \to \theta$ $(n \to \infty)$; X is said to be weak uniformly rotund, if for $\forall x_n, y_n \in S$, such that $\|x_n + y_n\| \to 2$, then $x_n - y_n \xrightarrow{w} \theta$ $(n \to \infty)$; The conjugate space X^* is said to be *weak** uniformly rotund, if $\forall x_n^*, y_n^* \in S^*$, such that $\|x_n^* + y_n^*\| \to 2$, then $x_n^* - y_n^* \xrightarrow{w^*} \theta$; X is said to be uniformly rotund in weakly compact sets directions, if when $\forall x_n, y_n \in S$ such that $\|x_n + y_n\| \to 2$, and $x_n - y_n \xrightarrow{w} z$ $(n \to \infty)$ then $z = \theta$.

Theorem 2　Let X be a Banach space, denote M, M^* the bounded nonempty subset in X and in X^* respectively. Let $\widetilde{A}$ be the collection of all norm closed and bounded nonempty subsets of $X\backslash\{\theta\}$, $\widetilde{B}$ be the collection of all weakly closed and bounded nonempty subsets of $X\backslash\{\theta\}$, $\widetilde{C}^*$ be the collection of all *weak** closed and bounded nonempty subsets of $X\backslash\{\theta\}$, and $\widetilde{D}$ be the collection of all norm compact nonempty subsets of $X\backslash\{\theta\}$. Then we have:

1) $\forall A\in\widetilde{A}, \eta(A,M)>0$ is equivalent to X is uniformly rotund;

2) $\forall B\in\widetilde{B}$, $\eta(B,M)>0$ is equivalent to X is weak uniformly rotund;

3) $\forall C^*\in\widetilde{C}^*, \eta(C^*,M^*)>0$ is equivalent to X^* is *weak** uniformly rotund;

4) $\forall D\in\widetilde{D}, \eta(D,M)>0$ is equivalent to X is uniformly rotund in weakly compact sets directions.

Proof　For the proof of 1) and 4), please check[5]. We now prove 2).

2) Assume $\forall B\in\widetilde{B}, \eta(B,M)>0$, but X is not weak uniformly rotund, which means that $\exists\varepsilon_0>0, f_0\in S^*$ and the subsequence $\{x_{n_k}\}$, $\{y_{n_k}\}$ of $\{x_n\}$, $\{y_n\}$, such that $\|x_{n_k}+y_{n_k}\|\to 2$, but $|f_0(x_{n_k})-f_0(y_{n_k})|\geqslant\varepsilon_0$, for $\forall k$. Noticing that $x_{n_k}, y_{n_k}\in S$, so $\|x_{n_k}\|=\|y_{n_k}\|=1$. For $\forall k$, we have

$$\varepsilon_0\leqslant|f_0(x_{n_k})-f_0(y_{n_k})|=|f_0(x_{n_k}-y_{n_k})|\leqslant\|f_0\|\|x_{n_k}-y_{n_k}\|=\|x_{n_k}-y_{n_k}\|\leqslant 2, \tag{8}$$

Denote B_0 as the weak closure of the set $\{z_{n_k}\mid z_{n_k}=x_{n_k}-y_{n_k}\}$, then it is clear that B_0 is a weakly closed and bounded nonempty subset of $X\backslash\{\theta\}$; Denote $M_0=\{x_{n_k}\}$, obviously M_0 is a bounded subset of X. From (8) we obtain:

$$0\leqslant 4\left(\frac{1}{2}\|x_{n_k}\|^2+\frac{1}{2}\|y_{n_k}\|^2-\left\|\frac{x_{n_k}+y_{n_k}}{2}\right\|^2\right)=2[\|x_{n_k}\|^2+\|y_{n_k}\|^2]-\|x_{n_k}+y_{n_k}\|^2\to 0,$$

and

$$\eta(B_0,M_0)=\inf_{z\in B_0}\left\{\inf_{\substack{x\in M_0\\ y\in X\\ x-y=z}}\left[\frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2-\left\|\frac{x+y}{2}\right\|^2\right]\right\}=0,$$

which contradicts to the assumption. So X is weak uniformly rotund.

On the other hand, if X is weak uniformly rotund, but the assumption $\forall B\in\widetilde{B}$, $\eta(B,M)>0$ does not hold, which means $\exists B_0\in\widetilde{B}$, and the bounded nonempty subset $M_0\subset X$, such that $\eta(B_0,M_0)=0$. It follows that there exists $z_n\in B_0, x_n\in M_0, y_n=x_n-z_n$, such that

$$2(\|x_n\|^2+\|y_n\|^2)-\|x_n+y_n\|^2=4\left(\frac{1}{2}\|x_n\|^2+\frac{1}{2}\|y_n\|^2-\left\|\frac{x_n+y_n}{2}\right\|^2\right)\to 0.$$

However $x_n-y_n=z_n\in B_0\subset X\backslash\{\theta\}$, which leads to that

$$\exists\varepsilon_0>0, \text{for } \forall f\in X^*, |f(x_n-y_n)|=|f(z_n)|\geqslant\varepsilon_0,$$

then we get the fact that $\{x_n-y_n\}=\{z_n\}$ do not convergent weakly to θ, hence by $2(\|x_n\|^2+\|y_n\|^2)-\|x_n+y_n\|^2\to 0$, we can not deduce $x_n-y_n=z_n\xrightarrow{w}\theta$ by the results in[3], which is a contradiction. So the proof of 2) is completed.

Employing the same technique as the proof in 2), the proof of 3) can be completed.

Theorem 3　Let $F(x)$ be the duality mapping in Banach Space X, $\eta(A,M)$ is the η-Rotundity modulus on the subset A of X concerning the bounded subset M. Here $\widetilde{A}, \widetilde{B}, \widetilde{D}$ is the same as we mentioned in Theorem2. M and M^* are the bounded subset of X and X^* respectively. Then we have:

1) $F(x)$ is uniformly continuous on any subset of X is equivalent to $\forall A\in\widetilde{A}, \eta(A,M)>0$;

2) $F(x)$ is weakly continuous on $X\Leftrightarrow\forall B\in\widetilde{B}, \eta(B,M)>0$;

3) $F(x)$ is *weak** continuous on $X^*\Leftrightarrow\forall C^*\in\widetilde{C}^*, \eta(C^*,M^*)>0$.

Proof　$F(x)$ is uniformly continuous on any subset of X is equivalent to X is uniformly rotund; $F(x)$ is weakly continuous on X is equivalent to X is weakly uniformly rotund; $F(x)$ is *weak** continuous on X^* is equivalent to X^* is *weak** uniformly rotund. Then by Theorem2, it is proved.

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